

CATEGORY EQUIVALENCES INVOLVING GRADED MODULES OVER QUOTIENTS OF WEIGHTED PATH ALGEBRAS

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ABSTRACT. Let k be a field, Q a finite directed graph, and kQ its path algebra. Make kQ an \mathbb{N} -graded algebra by assigning each arrow a positive degree. Let I be a homogeneous ideal in kQ and write $A = kQ/I$. Let $\text{QGr } A$ denote the quotient of the category of graded right A -modules modulo the Serre subcategory consisting of those graded modules that are the sum of their finite dimensional submodules. This paper shows there is a finite directed graph Q' with all its arrows placed in degree 1 and a homogeneous ideal $I' \subset kQ'$ such that $\text{QGr } A \equiv \text{QGr } kQ'/I'$. This is an extension of a result obtained by the author and Gautam Sisodia in [1].

1. INTRODUCTION

1.1. In noncommutative projective geometry, there seems to be a consensus that being generated in degree 1 is “good.”

For example, consider Serre’s Theorem: If A is a locally finite commutative graded k -algebra generated in degree 1, then $\text{QGr } A \equiv \text{Qcoh}(\text{Proj } A)$. Serre’s Theorem can fail if the algebra is not generated in degree 1, a counterexample being the polynomial algebra $k[x, y]$ with $\deg x = 1$ and $\deg y = 2$.

Another nice theorem that uses generation in degree 1 is Verevkin’s result about the equivalence

$$\text{QGr } A \equiv \text{QGr } A^{(d)}$$

where $A^{(d)}$ is the d -th Veronese subalgebra of A [3].

Given a graded algebra A , is it possible to find a graded algebra A' generated in degree one such that

$$\text{QGr } A \equiv \text{QGr } A'?$$

In [1] it was shown that the answer is yes when A is a path algebra or a monomial algebra. This article extends these results to include the case where A is any quotient of a path algebra by a finitely generated homogeneous ideal.

Lets consider the example with the commutative polynomial algebra $A = k[x, y]$ where $\deg x = 1$ and $\deg y = 2$. A is the quotient of the path algebra

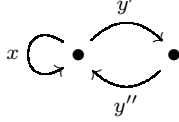
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kQ modulo the ideal $I = (xy - yx)$ where Q is the quiver



Let Q' be the quiver:



and give kQ' the grading where all arrows have degree 1. It is shown in [1] that $\text{QGr } kQ \equiv \text{QGr } kQ'$. That is, the noncommutative projective schemes $\text{Proj}_{nc} kQ$ and $\text{Proj}_{nc} kQ'$ are isomorphic.

The scheme $\text{Proj}_{nc} k[x, y]$ is a “closed subscheme” of $\text{Proj}_{nc} kQ$ defined by the ideal $I = (xy - yx)$. Since $\text{Proj}_{nc} kQ \cong \text{Proj}_{nc} kQ'$, the space $\text{Proj}_{nc} k[x, y]$ should correspond to some “closed subscheme” of $\text{Proj}_{nc} kQ'$.

One guess might be that $\text{Proj}_{nc} k[x, y]$ corresponds to the closed subscheme of $\text{Proj}_{nc} kQ'$ cut out by the ideal $I' = (xy'y'' - y'y''x)$. The methods of this paper show this is true. More explicitly, the main result shows

$$\text{QGr } k[x, y] \equiv \text{QGr } kQ'/I'.$$

This equivalence is rather interesting. The algebra $k[x, y]$ is a connected Noetherian domain while kQ'/I' is none of these. However, kQ'/I' is generated in degree 1. Thus, in trying to understand $\text{QGr } k[x, y]$, one can use whichever algebra is most suited to the question at hand.

The principal result of this paper is:

Theorem 1.1. *Let Q be a weighted quiver and I a finitely generated homogeneous ideal in kQ . There is a quiver Q' with all arrows having degree 1, a finitely generated homogeneous ideal $I' \subset kQ'$, and an equivalence of categories*

$$F : \text{QGr } kQ/I \equiv \text{QGr } kQ'/I'$$

which respects shifting. That is, $F(\mathcal{M}(1)) \cong F(\mathcal{M})(1)$ for all $\mathcal{M} \in \text{QGr } kQ/I$.

1.2. Notation and definitions. Throughout, $Q = (Q_0, Q_1, s, t)$ will always denote a finite quiver, i.e., a finite directed graph. The set Q_0 is called the vertex set, Q_1 the arrow set and $s, t : Q_1 \rightarrow Q_0$ will be the source and target maps respectively. Given a field k , the path algebra kQ is the algebra with basis consisting of all paths in Q , including a trivial path e_v at each vertex v .

Given two paths $p = a_1 \cdots a_n$ and $q = b_1 \cdots b_m$, the product pq is the path $a_1 \cdots a_n b_1 \cdots b_m$ if $t(a_n) = s(b_1)$ and is zero otherwise.

Call the pair (Q, \deg) a *weighted quiver* if Q is a finite quiver and $\deg : Q_1 \rightarrow \mathbb{N}_{>0}$. Usually, the \deg part of the notation (Q, \deg) will be dropped.

A weighted quiver determines an \mathbb{N} -graded path algebra kQ where the degree of the arrow a is $\deg(a)$ and the trivial paths have degree zero. The term *weighted path algebra* will mean the path algebra of a weighted quiver. The term *path algebra* will always mean the arrows have degree 1.

Given an \mathbb{N} -graded k -algebra A , $\text{Gr } A$ will denote the category of \mathbb{Z} -graded right A modules with degree preserving homomorphisms. $\text{Fdim } A$ will denote the localizing subcategory of $\text{Gr } A$ consisting of all graded modules which are the sum of their finite-dimensional submodules. The quotient of $\text{Gr } A$ by $\text{Fdim } A$ is denoted $\text{QGr } A$ and the canonical quotient functor will be denoted

$$\pi^* : \text{Gr } A \rightarrow \text{QGr } A.$$

The functor π^* is exact and the subcategory $\text{Fdim } A$ is localizing, that is, π^* has a right adjoint which will be denoted π_* .

2. THE CATEGORY OF GRADED REPRESENTATIONS WITH RELATIONS.

Associated to a weighted quiver Q is the category of graded representations $\text{GrRep } Q$. A graded representation is the data $M = (M_v, M_a)$ where for each vertex v , M_v is a \mathbb{Z} -graded vector space over k (k is in degree zero) and for each arrow a , $M_a : M_{s(a)} \rightarrow M_{t(a)}$ is a degree $\deg(a)$ linear map.

A morphism $\varphi : M \rightarrow N$ is a collection of degree 0 linear maps $\varphi_v : M_v \rightarrow N_v$ for each vertex v such that for each arrow $a \in Q_1$, the diagram

$$\begin{array}{ccc} M_{s(a)} & \xrightarrow{M_a} & M_{t(a)} \\ \varphi_{s(a)} \downarrow & & \downarrow \varphi_{t(a)} \\ N_{s(a)} & \xrightarrow{N_a} & N_{t(a)} \end{array}$$

commutes.

The categories $\text{Gr } kQ$ and $\text{GrRep } Q$ are equivalent. An explicit equivalence is given by sending a graded module M to the data (Me_v, M_a) where $M_a : Me_{s(a)} \rightarrow Me_{t(a)}$ is the degree $\deg(a)$ linear map induced by the action of a .

If $p = a_1 \cdots a_m$ is a path in Q , then given any graded representation (M_v, M_a) , p determines a degree $\deg(p)$ linear map $M_p : M_{s(a_1)} \rightarrow M_{t(a_m)}$ which is the composition

$$M_p = M_{a_m} \circ \cdots \circ M_{a_1}.$$

Given a linear combination $\rho = \sum \alpha_i p_i$, where $\alpha_i \in k$ and the p_i are paths in Q with the same source and target, we get a linear map

$$M_\rho = \sum \alpha_i M_{p_i}.$$

Let $A = kQ/I$ be a weighted path algebra modulo an ideal I generated by a finite number of homogeneous elements. Because of the idempotents e_v , we can write

$$I = (\rho_1, \dots, \rho_n)$$

where ρ_i is a linear combination of paths of the same degree all of which have the same source and target.

Let $\text{GrRep}(Q, \rho_1, \dots, \rho_n)$ denote the full subcategory of $\text{GrRep } Q$ consisting of all the graded representations (M_v, M_a) such that $M_{\rho_i} = 0$ for

all $i = 1, \dots, n$. The equivalence $\text{Gr } kQ \equiv \text{GrRep } Q$ induces an equivalence $\text{Gr } kQ/I \equiv \text{GrRep}(Q, \rho_1, \dots, \rho_n)$. From now on, the categories $\text{Gr } kQ/I$ and $\text{GrRep}(Q, \rho_1, \dots, \rho_n)$ will be identified.

3. PROOF OF THEOREM 1.1

3.1. The proof of Theorem 1.1 follows section 3 in [1] very closely. The details, with the appropriate modifications for the more general case, are reproduced here for convenience of the reader.

Given a weighted quiver Q , define the *weight discrepancy* to be the non negative integer

$$D(Q) := \left(\sum_{a \in Q_1} \deg(a) \right) - |Q_1|.$$

Note that $D(Q) = 0$ if and only if each arrow in Q has degree 1. The proof of Theorem 1.1 will be based on induction on $D(Q)$.

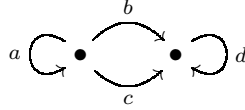
Let Q be a weighted quiver and suppose b is an arrow with $\deg(b) > 1$. Define a new quiver Q' from Q by declaring

$$\begin{aligned} Q'_0 &:= Q_0 \sqcup \{z\} \\ Q'_1 &:= (Q_1 \setminus \{b\}) \sqcup \{b' : s(b) \rightarrow z, b'' : z \rightarrow t(b)\}. \end{aligned}$$

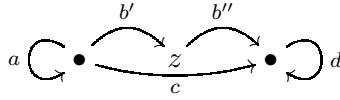
Make Q' a weighted quiver by letting each arrow in $Q'_1 \setminus \{b', b''\}$ have the same degree as it had in Q_1 and letting $\deg(b') = 1$ and $\deg(b'') = \deg(b) - 1$. From the construction of Q' it follows that

$$D(Q') = D(Q) - 1.$$

Example 3.1. Let Q be the quiver



with $\deg(b) > 1$. The associated quiver Q' is



with $\deg(b') = 1$ and $\deg(b'') = \deg(b) - 1$.

Let Q be a weighted quiver and Q' the associated quiver constructed above. Given a path $p = a_1 \cdots a_m$ in Q , let $f(p)$ be the path in Q' which is obtained by replacing every occurrence of b with $b'b''$ while leaving the path unchanged if there is no occurrence of b . For the quiver in example 3.1,

$$f(a^2bd) = a^2b'b''d$$

while

$$f(acd) = acd.$$

As $\deg(b'b'') = \deg(b)$, the map f preserves the degree of paths. Hence, f determines a graded k -linear map $f : kQ \rightarrow kQ'$ which can be seen to respect multiplication.

3.2. Let Q be a weighted quiver and Q' the associated quiver as in section 3.1. Given a graded representation $M \in \text{Gr } kQ$, let $F(M)$ be the following graded representation in $\text{Gr } kQ'$:

For the vertices;

- $F(M)_v := M_v$ for all $v \in Q'_1 \setminus \{z\}$,
- $F(M)_z := M_{s(b)}(-1)$,

while for the arrows;

- $F(M)_a := M_a$ for all $a \in Q'_1 \setminus \{b', b''\}$,
- $F(M)_{b'} := \text{id} : M_{s(b)} \rightarrow M_{s(b)}(-1)$ considered a linear map of degree 1,
- $F(M)_{b''} := M_b : M_{s(b)}(-1) \rightarrow M_{t(b)}$ considered a linear map of degree $\deg(b) - 1$.

Given a morphism $\varphi : M \rightarrow M'$ in $\text{Gr } kQ$, define $F(\varphi) : F(M) \rightarrow F(M')$ by

- $F(\varphi)_v := \varphi_v$ for all $v \in Q'_0 \setminus \{z\} = Q_0$, and
- $F(\varphi)_z := \varphi_{s(b)}(-1) : M_{s(b)}(-1) \rightarrow M'_{s(b)}(-1)$.

It is shown in [1] that $F : \text{Gr } kQ \rightarrow \text{Gr } kQ'$ is an exact functor for which

$$F(M(1)) \cong F(M)(1).$$

Let $p = a_1 \cdots a_m$ be a path in Q and $f(p)$ the associated path in Q' . From the definition of the functor F ,

$$F(M)_{f(p)} = M_p.$$

To see this, note $f(p) = f(a_1) \cdots f(a_m)$ so

$$F(M)_{f(p)} = F(M)_{f(a_m)} \cdots F(M)_{f(a_1)}.$$

If $a_i \neq b$, then $f(a_i) = a_i$ and thus $F(M)_{f(a_i)} = F(M)_{a_i} = M_{a_i}$. If $a_i = b$, then $f(a_i) = b'b''$ and thus $F(M)_{f(a_i)} = F(M)_{b'b''} = F(M)_{b''}F(M)_{b'} = M_b \circ \text{id} = M_b$. Hence, if $\rho = \sum \alpha_i p_i$ is a linear combination of paths with the same source and target, then

$$F(M)_{f(\rho)} = \sum \alpha_i F(M)_{f(p_i)} = \sum \alpha_i M_{p_i} = M_\rho.$$

Let $I = (\rho_1, \dots, \rho_n) \subset kQ$ be a homogeneous ideal. As before,

$$\rho_i = \sum_{j=1}^m \alpha_j p_j$$

is a linear combination of paths of the same degree such that $s(p_j) = s(p_{j'})$ and $t(p_j) = t(p_{j'})$ for all pairs (j, j') .

Suppose $M \in \text{Gr } kQ/I$. For all $\rho_i \in I$, $M_{\rho_i} = 0$. Hence, for the representation $F(M)$, $F(M)_{f(\rho_i)} = M_{\rho_i} = 0$ which implies $F(M) \in \text{Gr } kQ'/I'$ where I' is the ideal

$$I' = (f(\rho_1), \dots, f(\rho_n)).$$

Therefore, the functor $F : \text{Gr } kQ \rightarrow \text{Gr } kQ'$ induces a functor $F : \text{Gr } kQ/I \rightarrow \text{Gr } kQ'/I'$.

Let N be a representation of kQ' . Define $G(N)$ to be the following representation of kQ :

For the vertices,

- $G(N)_v := N_v$ for all vertices $v \in Q_0 = Q'_0 \setminus \{z\}$,

while for the arrows

- $G(N)_a := N_a$ for all $a \in Q_1 \setminus \{b\}$, and
- $G(N)_b := N_{b''} \circ N_{b'}$ which is a linear map of degree $\deg(b''b') = \deg(b)$.

Given a morphism $\psi : N \rightarrow N'$ in $\text{Gr } kQ'$, define $G(\psi) : G(N) \rightarrow G(N')$ by

- $G(\psi)_v := \psi_v$ for all $v \in Q_0 = Q'_0 \setminus \{z\}$.

G is a functor $\text{Gr } kQ' \rightarrow \text{Gr } kQ$.

Let N be a representation in $\text{Gr } kQ'$ and $p = a_1 \cdots a_m$ a path in Q . Since $G(N)_b = N_{b''}N_{b'}$ and $G(N)_a = N_a$ for $a \in Q_1 \setminus \{b\}$, it follows that

$$G(N)_p = N_{f(p)}$$

and more generally,

$$G(N)_\rho = N_{f(\rho)}$$

for any linear combination of paths with the same source and target. Hence, if N is a representation in $\text{Gr } kQ'/I'$, then for all $\rho_i \in I$,

$$G(N)_{\rho_i} = N_{f(\rho_i)} = 0.$$

Hence, the functor $G : \text{Gr } kQ' \rightarrow \text{Gr } kQ$ induces a functor $G : \text{Gr } kQ'/I' \rightarrow \text{Gr } kQ/I$.

From the definitions of F and G , it can be seen that $GF = \text{id}_{\text{Gr } kQ/I}$.

Let $N \in \text{Gr } kQ'/I'$, then the module $FG(N)$ is given by the data

- $FG(N)_v = N_v$ for $v \in Q'_0 \setminus \{z\}$,
- $FG(N)_z = N_{s(b)}(-1)$,
- $FG(N)_a = N_a$ for all $a \in Q'_1 \setminus \{b', b''\}$,
- $FG(N)_{b'} = \text{id} : N_{s(b)} \rightarrow N_{s(b)}(-1)$ considered a degree one linear map,
- $FG(N)_{b''} = N_{b''} \circ N_{b'} : N_{s(b)}(-1) \rightarrow N_{t(b)}$.

For each $N \in \text{Gr } kQ'/I'$, define $\epsilon_N : FG(N) \rightarrow N$ by $(\epsilon_N)_v = \text{id}$ for $v \neq z$ and $(\epsilon_N)_z = N_{b'}$ considered as a degree zero map from $FG(N)_z = N_{s(b)}(-1) \rightarrow N_z$.

Proposition 3.2. *The assignment $N \mapsto \epsilon_N$ is a natural transformation $\epsilon : FG \rightarrow \text{id}_{\text{Gr } kQ'/I'}$. Let $\eta : \text{id}_{\text{Gr } kQ/I} \rightarrow GF$ be the identity natural transformation. Then F is left adjoint to G with unit η and counit ϵ .*

Proof. See Propositions 3.3 and 3.4 in [1]. \square

3.3. Let $\pi^* : \text{Gr } kQ'/I' \rightarrow \text{QGr } kQ'/I'$ be the canonical quotient functor and π_* its right adjoint. Let $\sigma : \text{id}_{\text{Gr } kQ'/I'} \rightarrow \pi_*\pi^*$ be the unit and $\tau : \pi^*\pi_* \rightarrow \text{id}_{\text{QGr } kQ'/I'}$ the counit of the adjoint pair (π^*, π_*) . Using the adjoint pair (F, G) , we get the adjoint pair $(\pi^*F, G\pi_*)$ where

- $G\sigma F \cdot \eta : \text{id}_{\text{Gr } kQ/I} \rightarrow G\pi_* \circ \pi^*F$ is the unit and
- $\tau \cdot \pi^*\epsilon\pi_* : \pi^*F \circ G\pi_* \rightarrow \text{id}_{\text{QGr } kQ'/I'}$ is the counit.

As π^* and F are exact so is π^*F .

Lemma 3.3. *The kernel of $\pi^*F : \text{Gr } kQ/I \rightarrow \text{QGr } kQ'/I'$ is*

$$\text{Ker } \pi^*F = \text{Fdim } kQ/I.$$

Proof. Same as the proof of Lemma 3.5 in [1]. \square

Proposition 3.4. *For every module $N \in \text{Gr } kQ'/I'$, $\pi^*(\epsilon_N)$ is an isomorphism.*

Proof. For each vertex $v \in Q'_0 \setminus \{z\}$, $\epsilon_N = \text{id}_{N_v}$. Hence, $(\text{Ker } \epsilon_N)_v$ and $(\text{Coker } \epsilon_N)_v$ are zero for all vertices $v \in Q'_0 \setminus \{z\}$. Hence, the modules $\text{Ker } \epsilon_N$ and $\text{Coker } \epsilon_N$ are supported only on the vertex z . Thus, every arrow acts trivially on $\text{Ker } \epsilon_N$ and $\text{Coker } \epsilon_N$ showing they are both in $\text{Fdim } kQ'/I'$. Hence, the map $\pi^*(\epsilon_N)$ is an isomorphism. \square

Theorem 3.5. *The functor $\pi^*F : \text{Gr } kQ/I \rightarrow \text{QGr } kQ'/I'$ induces an equivalence of categories*

$$\text{QGr } kQ/I \equiv \text{QGr } kQ'/I'.$$

Proof. As F and π^* preserve shifting, π^*F preserves shifting. The functor π^*F is an exact functor with a right adjoint $G\pi_*$. For every object $\mathcal{N} \in \text{QGr } kQ'/I'$, the map $\pi^*(\epsilon_{\pi_*\mathcal{N}})$ is an isomorphism by Proposition 3.4. By [2, Prop. 4.3, pg. 176], the counit τ of the adjoint pair (π^*, π_*) is a natural isomorphism. Hence, the counit $\tau \cdot \pi^*\epsilon\pi_*$ is a natural isomorphism as

$$(\tau \cdot \pi^*\epsilon\pi_*)_{\mathcal{N}} = \tau_{\mathcal{N}} \circ \pi^*(\epsilon_{\pi_*\mathcal{N}})$$

is a composition of isomorphisms for all $\mathcal{N} \in \text{QGr } kQ'/I'$.

Thus, the right adjoint $G\pi_*$ is fully faithful. By [2, Theorem 4.9, pg. 180], π^*F induces an equivalence

$$\frac{\text{Gr } kQ/I}{\text{Ker } \pi^*F} \equiv \text{QGr } kQ'/I'$$

which preserves shifting. As $\text{Ker } \pi^*F = \text{Fdim } kQ/I$, the Theorem is proved. \square

3.4. Proof of Theorem 1.1. The proof of Theorem 1.1 now follows by induction on the weight discrepancy. If kQ/I is a quotient of a weighted path algebra for which $D(Q) = 0$, then every arrow in Q has degree 1 and there is nothing to prove. Suppose $D(Q) > 1$ and let b be an arrow in Q of degree greater than 1. Let Q' be the quiver obtained from Q by replacing the arrow b with two arrows as in Section 3.1 and I' the ideal obtained from the ideal I . By Theorem 3.5 there is an equivalence

$$\mathrm{QGr} \, kQ/I \equiv \mathrm{QGr} \, kQ'/I'.$$

which respects shifting. Since $D(Q') = D(Q) - 1$, we can find, by induction, a quiver Q'' with all arrows in degree 1 and a homogeneous ideal $I'' \subset kQ''$ such that

$$\mathrm{QGr} \, kQ'/I' \equiv \mathrm{QGr} \, kQ''/I''$$

where the equivalence respects shifting. Hence, $\mathrm{QGr} \, kQ/I \equiv \mathrm{QGr} \, kQ''/I''$ via an equivalence which respects shifting.

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